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Spring 1995, Math 110, Section 1

Second Midterm Exam

7 April, 1995
10:10-11:00

1. (Read all three parts of this question before answering.)

(a) (6 points) Assuming one knows how to define the determinant of a *square matrix*, define the determinant of a *linear operator* $\mathsf{T}: \mathsf{V} \to \mathsf{V}$ for V a finite-dimensional vector space.

(b) (6 points) What result must be proved to show that the determinant of T, as you defined it in (a), is well-defined?

(c) (8 points) Give the proof of the result referred to in (b). You may assume general results about the arithmetic of determinants, and about matrices of linear transformations.

2. (20 points) Below, I give a slightly reworded version of Gerschgorin's Disk Theorem, and a brief proof. At four spots in the proof, a statement is given in **bold type**. You are to give a brief justification of each of these statements below. (The assertions are preceded by marks [A] to [D]. Give each justification after the corresponding mark at the bottom of the page.) If you cannot justify some step, you can still give justifications for later steps assuming that step.

Gerschgorin's Disk Theorem. *Let* $A \in \mathsf{M}_{\mathsf{n}\times\mathsf{n}}(C)$. *For* $i = 1, \ldots, n$, *let* C_i *denote the closed disk in the complex plane centered at* A_{ii}, *and having radius equal to* $\Sigma_{j\neq i} |A_{ij}|$. *Then each eigenvalue of* A *lies in one of the disks* C_i.

Proof. Let λ be an eigenvalue of A. Then [A] **there exist complex numbers** $\boldsymbol{x_1, \ldots, x_n}$, **not all zero, such that for all** $\boldsymbol{i}$, $\boldsymbol{\Sigma_j A_{ij} x_j = \lambda x_i}$. Let k be chosen so that x_k has largest absolute value among the x_i. Observe that [B] $\boldsymbol{\lambda x_k - A_{kk} x_k = \Sigma_{j\neq k} A_{kj} x_j}$. Hence $|\lambda x_k - A_{kk} x_k| = |\Sigma_{j\neq k} A_{kj} x_j| \leq \Sigma_{j\neq k} |A_{kj}|\,|x_j|$. [C] **This last term is** $\boldsymbol{\leq \Sigma_{j\neq k} |A_{kj}|\,|x_k|}$.

Hence $|\lambda x_k - A_{kk} x_k| \leq \Sigma_{j\neq k} |A_{kj}|\,|x_k|$, or, factoring out $|x_k|$ on both sides, $|\lambda - A_{kk}|\,|x_k| \leq (\Sigma_{j\neq k} |A_{kj}|)\,|x_k|$. But [D] $\boldsymbol{|x_k| > 0}$. Hence we can divide this inequality by $|x_k|$, getting $|\lambda - A_{kk}| \leq \Sigma_{j\neq k} |A_{kj}|$, which says that λ lies in the disk C_k of radius $\Sigma_{j\neq k} |A_{kj}|$ about the point A_{kk}, as claimed. □

3. Let T be a linear operator on a vector space V.

(a) (6 points) Define what is meant by a T-invariant subspace of V.

(b) (10 points) Suppose V is finite-dimensional, of dimension n, and m is an integer $\leq n$. Show that if V has an ordered basis β such that $[\mathsf{T}]_\beta$ has the form $\begin{pmatrix} A & B \\ O & C \end{pmatrix}$, where A is $m \times m$, then V has an m-dimensional T-invariant subspace. (This is really an "if and only if" result, but to save time I am just asking you to prove this direction.)

4. Let A be an $n \times n$ matrix over a field F.

(a) (6 points) Define what is meant by the "characteristic polynomial of A".

(b) (6 points) The Cayley-Hamilton Theorem says that A "satisfies" its characteristic polynomial. Say precisely what this means. (If you use the concept of "substituting" a matrix into a polynomial, say what you mean by this.)

(c) (10 points) Show that the subspace of $\mathsf{M}_{n \times n}(F)$ spanned by $I, A, A^2, A^3, \ldots$ has dimension $\leq n$.

5. Suppose U and V are finite-dimensional inner product spaces over F (the field of real or complex numbers), and let $\mathsf{T}: \mathsf{U} \to \mathsf{V}$ be a linear map.

(a) (15 points) Prove that there exists a map $\mathsf{T}^*: \mathsf{V} \to \mathsf{U}$ such that for all $x \in \mathsf{U}$, $y \in \mathsf{V}$ one has $\langle \mathsf{T}(x), y \rangle_\mathsf{V} = \langle x, \mathsf{T}^*(y) \rangle_\mathsf{U}$, where $\langle \cdot, \cdot \rangle_\mathsf{U}$ and $\langle \cdot, \cdot \rangle_\mathsf{V}$ denote the inner products of U and V respectively. (For the sake of time, I do *not* ask you to prove that T* is also linear.) This is a modified version of a result in the text; you may use any results actually proved in the text in proving it.

(b) (7 points) Prove that the map T* constructed in (a) also satisfies the equation $\langle x, \mathsf{T}(y) \rangle_\mathsf{V} = \langle \mathsf{T}^*(x), y \rangle_\mathsf{U}$ for all $x \in \mathsf{V}$, $y \in \mathsf{U}$.