FINAL EXAM
MATH 121A
SPRING 2000

[1] **4 points**

A sequence of of complex numbers $\{c_n\}$ is a Cauchy sequence if and only if . . .

(Complete the definition, but please don't complete it by writing only " . . . it satisfies the Cauchy criterion.")

[2] **4 points**

Let S be nonempty set of real numbers which is bounded above.
$u = \sup S$ if and only if . . .

[3] 4 points

Let $f(x)$ and $g(x)$ be two complex-valued functions on $[0, 1]$. The root-mean-square distance between f and g is . . .

[4] 4 points

Let $\{\mathbf{v}_n\}$ be a sequence of vectors in an inner product space V. $\{\mathbf{v}_n\}$ converges to $\mathbf{v} \in V$ if and only if . . .

[5] 4 points

What is Cauchy's Integral Formula? (Please, do not write the formula alone, without explication.)

[6] 6 points

Let C denote the unit circle centered at 0 in the complex plane, oriented counterclockwise.

Evaluate

$$\oint_C \tan(z)dz \ .$$

Evaluatc

$$\oint_C \cot(z)dz \ .$$

[7] **6 points**

Find the radius of convergence of $\sum_{n=1}^{\infty} 2nz^{2n-1}$.

What does $\sum_{n=1}^{\infty} 2nz^{2n-1}$ equal where it converges?

[8] **6 points**

Expand $f(z) = \frac{\sin(z)}{z^4}$ in Laurent series about $z = 0$.

What is the residue at $z = 0$ of $\frac{\sin(z)}{z^4}$?

[9] 8 points

Let $\mathbf{x}$ be a vector in $\mathbb{C}^n$, and let $P_M(\mathbf{x})$ denote the projection of $\mathbf{x}$ onto a subspace $M \subset \mathbb{C}^n$.

Prove that $\|\mathbf{x} - P_M(\mathbf{x})\| \leq \|\mathbf{x} - \mathbf{m}\|$ for all $\mathbf{m} \in M$. In other words, prove that "the projection of $\mathbf{x}$ onto M is the vector in M that is closest to $\mathbf{x}$."

[10] 8 points

State the Spectral Theorem for operators on $\mathbb{C}^n$.

[11] **8 points**

Solve

$$y''(t) + y(t) = e^{-t} \quad ; \quad y(0) = 0,\ y'(0) = 0.$$

(Beware! This problems differs slightly from the one you may have seen in the list of possible exam problems.)

[12] **8 points**

By contour integration, evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} dx \; .$$

[13] **10 points**
Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} .$$

Write A as $U \Lambda U^\dagger$, where Λ is a diagonal matrix and U is a unitary matrix.

[14] **10 points**

Choose **one** and write a very careful answer:

(i)

Let $\{a_j\}$ be a sequence of real numbers with $a_1 \leq a_2 \leq a_3 \leq \cdots$, and suppose $u = \sup\{a_j\} < \infty$. Prove that

$$\lim_{n \to \infty} a_j = u \ .$$

(ii)

Suppose $\{c_j\}$ converges to c. Prove that $\{|c_j|\}$ converges to $|c|$.

11

[15] 10 points

Let $f(x)$ be a square-integrable function on $\mathbb{R}$. Use the Fourier Integral to solve the heat equation on $\mathbb{R}$:

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= a^2\frac{\partial^2}{\partial x^2}u(x,t)\\ u(x,0) &= f(x)\ .\end{aligned}$$

You may use the fact that

$$\mathcal{F}^{-1}[f(\omega)g(\omega)] = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}[f] \star \mathcal{F}^{-1}[g]$$

and the following formula for the Fourier transform of a Gaussian function:

$$\mathcal{F}\left[e^{-x^2/2\sigma^2}\right](k) = \sigma e^{-k^2/2\sigma^{-2}}.$$

12